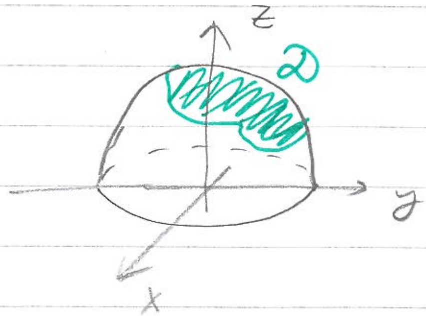
Lektion 21

6.26



Låt $\mathcal{D}$ vara integrationsområde i den övre delen av enhetsklottet.

I så fall

$$\iiint_{\mathcal{D}} 3z\,dx\,dy\,dz \le \iiint_{K} 3z\,dx\,dy\,dz$$

där K är den övre delen av enhetsklottet.

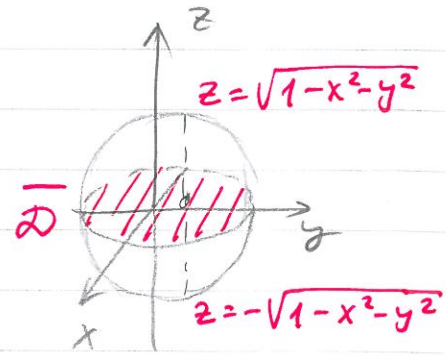
Men $z \le 1$ i $K \Rightarrow$

$$\iiint_{\mathcal{D}} 3z\,dx\,dy\,dz \le \iiint_{K} 3z\,dx\,dy\,dz \le \iiint_{K} 3\,dx\,dy\,dz =$$

$$= 3\,\text{Volym}(K) = \cancel{3} \cdot \frac{1}{2} \cdot \underbrace{\frac{4}{\cancel{3}}\pi \cdot 1^3}_{\text{hela klotets volym}} = 2\pi$$

$\Rightarrow \iiint_{\mathcal{D}} 3z\,dx\,dy\,dz \le 2\pi \Rightarrow$ varje svar $> 2\pi$ är inte rimligt.

6.27



a) Områdets projektion på xy-planet är enhetscirkelskivan $\overline{\mathcal{D}} = \{x^2+y^2 \le 1.\}$ När $(x,y) \in \overline{\mathcal{D}} \Rightarrow$

$-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2} \Rightarrow$ integralen blir

$$\iiint_{\mathcal{D}} (z^2-x^2-y^2)\,dx\,dy\,dz = \iint_{\bar{\mathcal{D}}} \left(\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (z^2-x^2-y^2)\,dz \right) dx\,dy =$$

$$= \iint_{\bar{\mathcal{D}}} \left[\frac{z^3}{3} - (x^2+y^2)z \right]_{z=-\sqrt{1-x^2-y^2}}^{z=\sqrt{1-x^2-y^2}} dx\,dy =$$

$$= \iint_{\bar{\mathcal{D}}} \left(\frac{2(1-x^2-y^2)^{3/2}}{3} - 2(x^2+y^2)\sqrt{1-x^2-y^2} \right) dx\,dy =$$

$$= \left[\begin{array}{l} \bar{\mathcal{D}} \text{ - cirkelskivan} \Rightarrow \text{byter mot polära koordinater} \\ \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \quad 0 \le r \le 1, \quad 0 \le \varphi \le 2\pi, \quad x^2+y^2 = r^2 \end{array} \right] =$$

$$= \int_0^{2\pi} \underbrace{\int_0^1 \left(\frac{2(1-r^2)^{3/2}}{3} - 2r^2\sqrt{1-r^2} \right) r\,dr}_{\text{konst map } \varphi}\, d\varphi =$$

$$= \left[\begin{array}{l} \text{variabelbyte } r^2 = t, \quad dt = 2r\,dr \\ 0 \le r \le 1 \Rightarrow 0 \le t \le 1 \end{array} \right] =$$

$$= \frac{2\pi}{2} \int_0^1 \left(\frac{2(1-t)^{3/2}}{3} - 2t\sqrt{1-t} \right) dt =$$

$$= \pi \int_0^1 \left(\frac{2}{3}(1-t)^{3/2} \underbrace{+ 2\overbrace{(1-t)\sqrt{1-t}}^{=(1-t)^{3/2}} - 2\sqrt{1-t}}_{= -2t\sqrt{1-t}} \right) dt$$

$$= \pi \int_0^1 \left(\frac{8}{3}(1-t)^{3/2} - 2(1-t)^{\frac{1}{2}} \right) dt =$$

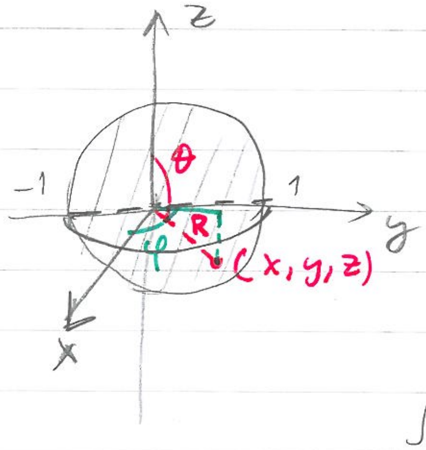
$$= \pi \left[-\frac{8}{3} \cdot \frac{2}{5}(1-t)^{5/2} + \frac{4}{3}(1-t)^{3/2} \right]_{t=0}^{t=1} =$$

$$= \pi \left(0 + \frac{16}{15} + 0 - \frac{4}{3} \right) = \pi \cdot \frac{16-20}{15} = \boxed{-\frac{4\pi}{15}}$$

b) Vi använder variabelbytet mot sfäriska (rymdpolära) koordinater:

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \qquad \left|\frac{d(x,y,z)}{d(r,\theta,\varphi)}\right| = r^2\sin\theta$$

$\Rightarrow x^2+y^2+z^2 = r^2$ (se boken s 291–292 för en bild)



Vi ser att

$0 \le \theta \le \pi$, $-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$, $0 \le r \le 1$.

Integralen blir

$$\iiint_D x e^{x^2+y^2+z^2}\, dx\,dy\,dz =$$

$$= \int_0^{\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_0^1 r\sin\theta\cos\varphi\, e^{r^2}\cdot \underbrace{r^2\sin\theta}_{=\left|\frac{d(x,y,z)}{d(r,\theta,\varphi)}\right|}\, dr\right)d\varphi\right)d\theta =$$

$$= \int_0^{\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\sin^2\theta\cos\varphi \int_0^1 r^2 e^{r^2}\cdot r\, dr\right)d\varphi\, d\theta = \otimes$$

$$\int_0^1 r^2 e^{r^2}\cdot r\,dr = \begin{bmatrix} r^2 = t, \; 0 \le r \le 1 \Rightarrow 0 \le t \le 1, \\ dt = 2r\,dr \end{bmatrix} =$$

$$= \frac{1}{2}\int_0^1 t e^t\, dt = \frac{1}{2}\left(\left[te^t\right]_0^1 - \left[e^t\right]_0^1\right) =$$

$$= \frac{1}{2}(e - e + 1) = \frac{1}{2} \quad \Rightarrow$$

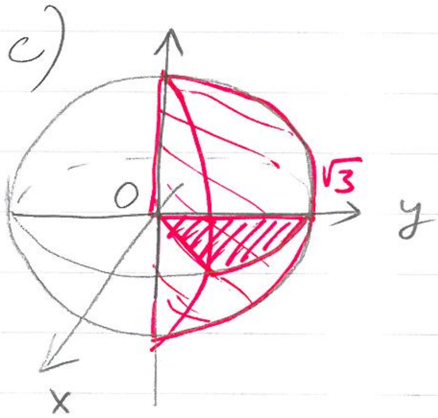
$$\otimes = \frac{1}{2}\int_0^{\pi}\left(\sin^2\theta \int_{-\frac{\pi}{2}}^{\pi/2} \cos\varphi\, d\varphi\right)d\theta =$$

$$= \frac{1}{2} \int_0^{\pi} \left(\sin^2\theta \cdot \left[\sin\varphi \right]_{\varphi=-\frac{\pi}{2}}^{\varphi=\frac{\pi}{2}} d\theta \right) =$$

$$= \frac{1}{2} \int_0^{\pi} \frac{1 - \cos 2\theta}{\cancel{2}} \cdot \cancel{2} \, d\theta =$$

$$= \frac{1}{2} \left(\pi - \underbrace{\cancel{\left[\frac{\sin 2\theta}{2} \right]_{\theta=0}^{\theta=\pi}}}_{=0} \right) = \boxed{\frac{\pi}{2}}$$

c)



Området är en åttonde del av klotet $x^2+y^2+z^2 \leq 3$

I rymdpolära koordinater kan vi skriva det som

$$0 \leq r \leq \sqrt{3} \qquad \begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r \cos\theta \end{cases}$$

$$\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}$$

$$0 \leq \theta \leq \pi$$

$$\left| \frac{d(x,y,z)}{d(r,\theta,\varphi)} \right| = r^2 \sin\theta$$

Integralen blir

$$\int_0^{\sqrt{3}} \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\int_0^{\pi} \underbrace{r \sin\theta \cos\varphi}_{=x} \cdot \underbrace{r^2 \sin\theta}_{=\left|\frac{d(x,y,z)}{d(r,\theta,\varphi)}\right|} d\theta \right) d\varphi \right) dr =$$

$$= \int_0^{\sqrt{3}} \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\int_0^{\pi} r^3 \sin^2\theta \cos\varphi \, d\theta \right) d\varphi \right) dr =$$

$$= \int_0^{\sqrt{3}} \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(r^3 \cos\varphi \underbrace{\int_0^{\pi} \frac{1 - \cos 2\theta}{2} d\theta}_{=\frac{\pi}{2}} \right) d\varphi \right) dr =$$

$$= \int_0^{\sqrt{3}} \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(r^3 \cos\varphi \cdot \frac{\pi}{2} \right) d\varphi \right) dr =$$

$$= \frac{\pi}{2} \int_0^{\sqrt{3}} \left(r^3 \left[\sin\varphi \right]_{\varphi = \frac{\pi}{4}}^{\varphi = \frac{\pi}{2}} \right) dr =$$

$$= \frac{\pi}{2} \int_0^{\sqrt{3}} r^3 \left(1 - \frac{\sqrt{2}}{2} \right) dr = \frac{\pi}{2} \left(1 - \frac{\sqrt{2}}{2} \right) \left[\frac{r^4}{4} \right]_{r=0}^{r=\sqrt{3}} =$$

$$= \frac{\pi}{2} \left(1 - \frac{\sqrt{2}}{2} \right) \cdot \frac{9}{4} = \boxed{\frac{9\pi(2-\sqrt{2})}{16}}.$$

6.28

a) Eftersom D bestäms av dubbelolikheterna $0 \le x+y+z \le 1$, $0 \le x+2y+3z \le 1$, $0 \le x+4y+9z \le 1$ är det praktiskt att göra variabelbytet

$$\begin{aligned} u &= x+y+z \\ v &= x+2y+3z \\ w &= x+4y+9z \end{aligned} \Rightarrow - \begin{cases} u = x+y+z \\ u-v = -y-2z \quad | \cdot 2 \\ v-w = -2y-6z \end{cases} \Leftrightarrow$$

$$2u-2v-v+w = -4z+6z \Rightarrow 2z = 2u-3v+w$$

$$\Leftrightarrow \begin{cases} z = u - \frac{3}{2}v + \frac{1}{2}w \\ y = -2z - u + v = -2u+3v-w-u+v = -3u+4v-w \\ x = u-y-z = \cancel{u}+3u-4v+w-\cancel{u}+\frac{3}{2}v-\frac{1}{2}w = \end{cases}$$

$$= 3u - \frac{5}{2}v + \frac{w}{2}$$

från 2a ekv.

från 1a ekv.

$$y - x - z = -3u+4v-w-3u+\frac{5}{2}v-\frac{w}{2}-u+\frac{3}{2}v-\frac{w}{2}$$

$$= -7u+8v-2w$$

$$\frac{d(x,y,z)}{d(u,v,w)} = \left(\frac{d(u,v,w)}{d(x,y,z)}\right)^{-1} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}^{-1} =$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix}^{-1} = \left(1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix}\right)^{-1} =$$

$$= (8-6)^{-1} = \frac{1}{2}$$

Vi ser att $\mathcal{D} = \{0 \leq u \leq 1,\ 0 \leq v \leq 1,\ 0 \leq w \leq 1\} \Rightarrow$

$$\iiint_{\mathcal{D}} (y-x-z)\,dx\,dy\,dz = \int_0^1 \left(\int_0^1 \left(\int_0^1 \left(-7u + 8v - 2w\right) \cdot \frac{1}{2}\,du\right) dv\right) dw$$

$$= \frac{1}{2}\int_0^1 \left(\int_0^1 \left[-\frac{7u^2}{2} + 8vu - 2wu\right]_{u=0}^{u=1} dv\right) dw =$$

$$= \frac{1}{2}\int_0^1 \left(\int_0^1 \left(-\frac{7}{2} + 8v - 2w\right) dv\right) dw =$$

$$= \frac{1}{2}\int_0^1 \left(\left[-\frac{7}{2}v + 4v^2 - 2wv\right]_{v=0}^{v=1}\right) dw =$$

$$= \frac{1}{2}\int_0^1 \left(\underbrace{-\frac{7}{2} + 4}_{=\frac{1}{2}} - 2w\right) dw =$$

$$= \frac{1}{2}\left[\frac{1}{2}w - w^2\right]_{w=0}^{w=1} = \boxed{-\frac{1}{4}}.$$

6.30

a) $\text{Area}(\mathcal{D}) = \iint_{\mathcal{D}} dx\,dy$. Eftersom

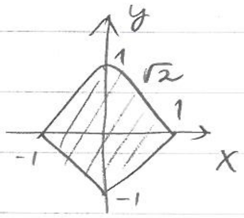
$\mathcal{D} = \{ |x+2y| + |3x-y| \le 1 \}$ vi gör ett variabelbyte

$$\begin{cases} x+2y = u \\ 3x - y = v \end{cases} \Rightarrow \left|\frac{d(x,y)}{d(u,v)}\right| = \left(\frac{d(u,v)}{d(x,y)}\right)^{-1} =$$

$$= \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}^{-1} = -\frac{1}{7}.$$

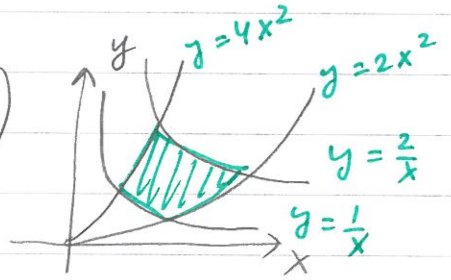
$$\iint_{\mathcal{D}} dx\,dy = \iint_{|u|+|v|\le 1} \left|\frac{d(x,y)}{d(u,v)}\right| du\,dv = \frac{1}{7} \iint_{|u|+|v|\le 1} du\,dv =$$

$$= \frac{1}{7} \text{Area}\Big(\underbrace{\{|u|+|v|\le 1\}}_{\text{kvadrat med sidan } \sqrt{2}}\Big) =$$



$$= \frac{1}{7}(\sqrt{2})^2 = \boxed{\frac{2}{7}}$$

b)



$$\text{Area}(\mathcal{D}) = \iint_{\mathcal{D}} dx\,dy,$$

Vi gör variabelbytet

$$\begin{cases} u = xy \\ v = \frac{y}{x^2} \end{cases} \Rightarrow \mathcal{D} = \{1 \le u \le 2;\ 2 \le v \le 4\}$$

$$\frac{d(x,y)}{d(u,v)} = \left(\frac{d(u,v)}{d(x,y)}\right)^{-1} = \begin{vmatrix} y & x \\ -\frac{2y}{x^3} & \frac{1}{x^2} \end{vmatrix}^{-1} = \frac{1}{\frac{y}{x^2} + \frac{2y}{x^2}} = \frac{1}{\frac{3y}{x^2}}$$

$$= \frac{1}{3v} \Rightarrow$$

$$\iint_{\mathcal{D}} dx\,dy = \int_1^2 \left(\int_2^4 \frac{1}{3v}\,dv \right) du = \int_1^2 \left[\frac{\ln v}{3} \right]_{v=2}^{v=4} du =$$

$$= \int_1^2 \left(\frac{\ln 4}{3} - \frac{\ln 2}{3}\right) du = \left(\frac{\ln 4}{3} - \frac{\ln 2}{3}\right) [u]_{u=1}^{u=2} =$$

$$= \frac{\ln 4}{3} - \frac{\ln 2}{3} = \frac{2\ln 2 - \ln 2}{3} = \frac{\ln 2}{3}$$

6.31

a) $V(\mathcal{D}) = \iiint_{\mathcal{D}} dx\,dy\,dz$.

Eftersom $\mathcal{D} = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\}$ byter mot rymdpolära koordinater (vi modifierar dem så att $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = r^2$) $\Rightarrow$

$$\begin{cases} x = a r \sin\theta \cos\varphi \\ y = b r \sin\theta \sin\varphi \\ z = c r \cos\theta \end{cases} \quad \begin{matrix} 0 \le r \le 1 \\ 0 \le \theta \le \pi \\ 0 \le \varphi \le 2\pi \end{matrix}$$

, i vilket fall

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \underbrace{(r\sin\theta\cos\varphi)^2 + (r\sin\theta\sin\varphi)^2 + (r\cos\theta)^2}_{\text{vanliga rymdpolära koord.}} = r^2.$$

Funktionaldeterminanten

$$\frac{d(x,y,z)}{d(r,\theta,\varphi)} = \underbrace{a\cdot b\cdot c}_{\text{då vi kan bryta a, b, c från 1a, 2a, 3e raden!}} \cdot \left\{\begin{matrix}\text{funktional-}\\\text{determinanten}\\\text{för vanliga}\\\text{rymdpolära}\end{matrix}\right\} = abc\,r^2\sin\theta$$

$$\iiint_{\mathcal{D}} dx\,dy\,dz = \int_0^1\left(\int_0^\pi\left(\int_0^{2\pi} abc\,r^2\sin\theta\,d\varphi\right)d\theta\right)dr =$$

$$= abc \int_0^1 \left(\int_0^{\pi} r^2 \sin\theta \cdot 2\pi \, d\theta \right) dr =$$

$$= 2\pi abc \int_0^1 r^2 \underbrace{\left[-\cos\theta\right]_{\theta=0}^{\theta=\pi}}_{=2} dr =$$

$$= 4\pi abc \cdot \left[\frac{r^3}{3}\right]_{r=0}^{r=1} = \boxed{\frac{4\pi abc}{3}}$$

b) $\text{Volym}(\mathcal{D}) = \iiint_{\mathcal{D}} dx\,dy\,dz$, där

$\mathcal{D} = \{3x^2 + 2y^2 + z^2 + 2xz - 2yz \leq 1\}$.

Vi kvadratkompletterar

$$3x^2 + 2y^2 + \underline{z^2 + 2z(x-y)} =$$

$$= (z + (x-y))^2 - (x-y)^2 + 3x^2 + 2y^2 =$$

$$= (z + x - y)^2 - x^2 + 2xy - y^2 + 3x^2 + 2y^2 =$$

$$= (x - y + z)^2 + \overbrace{2x^2}^{x^2+x^2} + \underline{2xy + y^2} =$$

$$= (x - y + z)^2 + (x+y)^2 + x^2$$

Om vi nu byter

$$\begin{cases} u = x \\ v = x + y \\ w = x - y + z \end{cases} \Rightarrow \mathcal{D} = \{u^2 + v^2 + w^2 \leq 1\} = \text{enhetsklot}$$

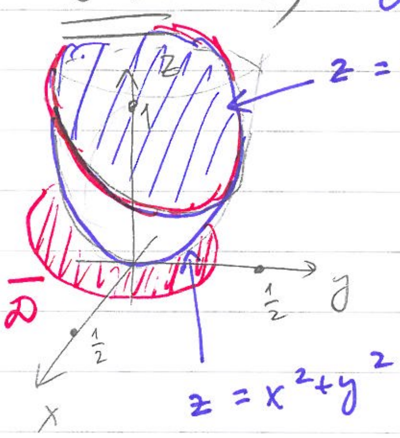
med volym $\frac{4}{3}\pi$

$$\frac{d(x,y,z)}{d(u,v,w)} = \left(\frac{d(u,v,w)}{d(x,y,z)}\right)^{-1} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix}^{-1} = 1 \Rightarrow$$

$$\Rightarrow \iiint_{\mathcal{D}} dx\,dy\,dz = \iiint_{u^2+v^2+w^2\le 1} \cdot 1 \, du\,dv\,dw =$$

$$= \boxed{\frac{4\pi}{3}} = \text{volym av enhetsklot.}$$

6.33 b) OBS! 6.33 a) är på nästa blad!



röd på bilden

skärningen mellan paraboloiden och planet beskrivs av ekvationssystem

$$\begin{cases} z = 1-2x-2y \\ z = x^2+y^2 \end{cases} \Leftrightarrow \begin{cases} x^2+y^2 = 1-2x-2y \\ z = x^2+y^2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x^2+2x+1+y^2+2y+1 = 3 \\ z = x^2+y^2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (x+1)^2+(y+1)^2 = 3 \\ z = x^2+y^2 \end{cases}$$

Vi ser att kroppens projektion på xy-planet är cirkeln $\overline{\mathcal{D}} = \{(x+1)^2+(y+1)^2 \le 3\}$, och att området kan skrivas som $\mathcal{D} = \{x^2+y^2 \le z \le 1-2x-2y\} \Rightarrow$

$$\text{Volym}(\mathcal{D}) = \iiint_{\mathcal{D}} dx\,dy\,dz = \iint_{\overline{\mathcal{D}}} \left(\int_{x^2+y^2}^{1-2x-2y} dz \right) dx\,dy =$$

$$= \iint_{\overline{\mathcal{D}}} \left(1-2x-2y-x^2-y^2\right) dx\,dy = \iint_{\overline{\mathcal{D}}} \left(3-(x+1)^2-(y+1)^2\right) dx\,dy =$$

$$= \left[\begin{array}{l} x = r\cos\varphi - 1 \\ y = r\sin\varphi - 1 \end{array} \Rightarrow (x+1)^2+(y+1)^2 = r^2 \le 3 \Rightarrow 0 \le r \le \sqrt{3},\ 0 \le \varphi \le 2\pi \quad \frac{d(x,y)}{d(r,\varphi)} = r \right] =$$

$$= \int_0^{2\pi} \left(\int_0^{\sqrt{3}} (3-r^2)\, r\, dr \right) d\varphi = \int_0^{2\pi} \left[\frac{3r^2}{2} - \frac{r^4}{4} \right]_{r=0}^{r=\sqrt{3}} d\varphi = 2\pi \cdot \left(\frac{9}{2} - \frac{9}{4} \right) =$$

$$= \boxed{\frac{9\pi}{2}}$$

6.33

a) $9x^2 + 4y^2 \le 36 \Leftrightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 \le 1$.

Observera att $|x| \le 2$, $|y| \le 3 \Rightarrow$

$z = 10 - x - y > 0$

$z = x + y - 10 < 0$

Området kan skrivas som

$$\mathcal{D} = \{ -(10-x-y) \le z \le 10-x-y \; ; \; (x,y) \in \overline{\mathcal{D}} \},$$

där $\overline{\mathcal{D}} = \{ \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 \le 1 \} \Rightarrow$

$$\text{Volym}(\mathcal{D}) = \iiint_{\mathcal{D}} dx\,dy\,dz =$$

$$= \iint_{\overline{\mathcal{D}}} \left(\int_{-(10-x-y)}^{10-x-y} dz \right) dx\,dy =$$

$$= \iint_{\overline{\mathcal{D}}} (20 - 2x - 2y)\,dx\,dy = \otimes$$

Gör variabelbytet $\begin{cases} x = 2r\cos\varphi \\ y = 3r\sin\varphi \end{cases}$,

så att $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = r^2 \le 1 \Rightarrow \begin{matrix} 0 \le r \le 1 \\ 0 \le \varphi \le 2\pi \end{matrix}$

$$\frac{d(x,y)}{d(r,\varphi)} = \begin{vmatrix} 2\cos\varphi & -2r\sin\varphi \\ 3\sin\varphi & 3r\cos\varphi \end{vmatrix} = 6r \Rightarrow$$

$$\otimes = 6\int_0^1 \left(\int_0^{2\pi} (20 - \underbrace{2r\cos\varphi - 2r\sin\varphi}_{\text{ger 0 vid integration}})\, r\,d\varphi \right) dr = 12\pi \int_0^1 20r\,dr =$$

$$= 12\pi \cdot 20 \cdot \left[\frac{r^2}{2}\right]_{r=0}^{r=1} = 120\pi$$

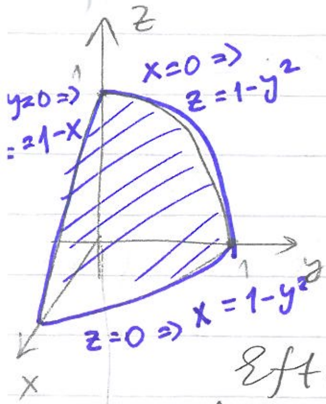
OBS! 6.33 b) är på s. 10.

6.34

a) För varje punkt $(x,y,z) \in \mathcal{D}$ har vi $z \geq 0$, samtidigt som z kan inte vara större än z som satisfierar ytans ekvation $x + y^2 + z = 1$ $\Rightarrow$

$$0 \leq z \leq 1 - x - y^2 \leq 1 \Rightarrow 0 \leq z \leq 1.$$

Likadant finner vi att



$$0 \leq x \leq 1 - y^2 - z^2 \leq 1 \Rightarrow 0 \leq x \leq 1 \quad \text{och att}$$

$$0 \leq y^2 \leq 1 - x - z \leq 1 \Rightarrow 0 \leq y^2 \leq 1 \Rightarrow$$

$$0 \leq y \leq 1$$

då y är positiv.

Eftersom $\mathcal{D}$ är en del av enhetskuben $\{0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1\}$ måste Volym $(\mathcal{D}) \leq 1$.

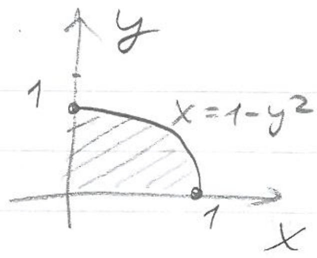
b) Eftersom $0 \leq x + y^2 + z \leq 1$, ser vi att mängdens projektion på xy-planet är

$$\overline{\mathcal{D}} = \{x \geq 0,\ y \geq 0,\ x + y^2 \leq 1\}.$$

När $(x,y) \in \overline{\mathcal{D}} \Rightarrow 0 \leq z \leq 1 - (x + y^2) \Rightarrow$

vi kan skriva $\mathcal{D} = \{0 \leq z \leq 1 - (x + y^2);\ (x,y) \in \overline{\mathcal{D}}\}$.

$$\text{Volym}(\mathcal{D}) = \iiint_{\mathcal{D}} dx\,dy\,dz = \iint_{\overline{\mathcal{D}}} \left(\int_0^{1-x-y^2} dz \right) dx\,dy =$$



$$= \iint_{\mathcal{D}} (1-x-y^2)\,dx\,dy =$$

$$= \left[\mathcal{D} = \{ 0 \le x \le 1-y^2,\ 0 \le y \le 1 \} \right] =$$

$$= \int_0^1 \left(\int_0^{1-y^2} (1-x-y^2)\,dx \right) dy =$$

$$= \int_0^1 \left[(1-y^2)x - \frac{x^2}{2} \right]_{x=0}^{x=1-y^2} dy =$$

$$= \int_0^1 \frac{(1-y^2)^2}{2}\,dy = \int_0^1 \frac{1-2y^2+y^4}{2}\,dy =$$

$$= \left[\frac{y}{2} - \frac{y^3}{3} + \frac{y^5}{10} \right]_{y=0}^{y=1} = \frac{1}{2} - \frac{1}{3} + \frac{1}{10} = \frac{30-20+6}{60} = \frac{16}{60} = \boxed{\frac{4}{15}}$$

Extra

6.32

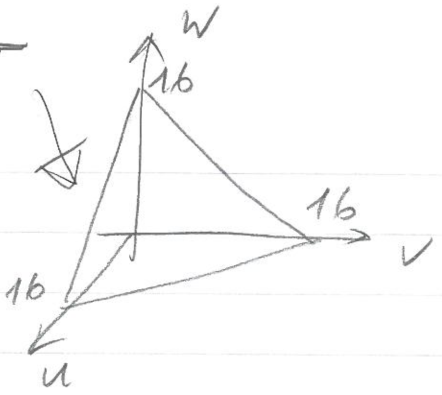
Vi inför nya koordinater

$$\begin{cases} 2x+y+z=u \\ x+2y+z=v \\ x+y+2z=w \end{cases} \Rightarrow 4x+4y+4z=u+v+w$$

så ekvationen för det fjärde planet kan skrivas som $u+v+w=16$

I u,v,w koordinater avgränsas vårt område av plan $u=0,\ v=0,\ w=0,\ u+v+w=16 \Rightarrow$

Detta är tetraedr T med volym



$$\frac{1}{3} \cdot \underbrace{16}_{\text{höjden}} \cdot \underbrace{\frac{1}{2} \cdot 16^2}_{\text{basytan}} =$$

$$= \frac{16^3}{6}.$$

Observera att $\dfrac{d(x,y,z)}{d(u,v,w)} = \left(\dfrac{d(u,v,w)}{d(x,y,z)}\right)^{-1} =$

$$= \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}^{-1} = \frac{1}{4}, \text{ då}$$

$$\left|\begin{array}{ccc|cc} 2 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 1 \end{array}\right| \rightsquigarrow 8+1+1-2-2-2 = 4$$

Slutligen

$$\iiint_{D} dx\,dy\,dz = \iiint_{T} \frac{d(x,y,z)}{d(u,v,w)}\, du\,dv\,dw =$$

$$= \frac{1}{4} \operatorname{Volym}(T) = \frac{1}{4} \cdot \frac{16^3}{6} = \frac{4 \cdot 16^2}{6} =$$

$$= \frac{2 \cdot 4 \cdot 4 \cdot 16}{3} = \frac{8 \cdot 64}{3} = \boxed{\frac{512}{3}}$$